Warm-ups

① Find a vector parallel to the plane $3x - y + 7z = 2024$.

② Prove the Cauchy-Schwarz Inequality: For all $v, w \in \mathbb{R}^n$, $|v \cdot w| \le |v|\,|w|$.

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① e.g. $(1,10,1)$ * Normal to plane: $\langle 3,-1,7\rangle$
any vector perp to that would be parallel to the plane.

$(2,-1,-1)$

$(0,0,0)$

$(1,3,0)$

e.g. $(1,10,1)\cdot(3,-1,7) = 3-10+7=0.$

eg. $(2,-1,-1)\cdot(3,(-1),7) = 6+1-7=0$

$(0,0,0)\cdot(3,-1,7) = 0$

$(1,3,0)\cdot(3,-1,7) = 3-3=0.$

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② Proof: $|v\cdot w| = \big||v|\,|w|\cos\theta\big| = |v|\,|w|\,|\cos\theta| \le |v|\,|w|.$ ∎

geometric formula of $\cdot$

New topic from linear algebra – determinants of matrices.

A matrix is a rectangular array of numbers:

$$\begin{pmatrix}2&3\\7&-1\end{pmatrix},\quad \begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \begin{pmatrix}2+i&7\\4&3\end{pmatrix},\quad \begin{pmatrix}1&2&0\\2&0&0\\3&1&-4\end{pmatrix}$$

Determinant of a matrix A:

2×2 $$\det\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{vmatrix}a&b\\c&d\end{vmatrix} = ad-bc$$

eg $\begin{pmatrix}2&3\\7&-1\end{pmatrix}$: $\det\begin{pmatrix}2&3\\7&-1\end{pmatrix} = 2(-1) - 3\cdot 7 = -2-21 = \boxed{-23}.$

$$\det\begin{pmatrix}1&758\\0&45\end{pmatrix} = 1\cdot 45 - 758\cdot 0 = \boxed{45}.$$

3x3 or n x n.

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{vmatrix} =$$

① pick row or column ← try for simple

② imagine the ± grid

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - \\ + & - & + \end{pmatrix}$$

③ For that one row or column, add according to the signs, and multiply each entry by the corresponding minor determinant.

$$= -2\begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{vmatrix} - 0\begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{vmatrix}$$

$$= -2\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ 0 & -1 \end{vmatrix}$$

$$= -2\left(0\cdot 2-(-1)(-1)\right) + 1\left(3\cdot 2 - 1(-1)\right) - 0\left(3(-1)-1\cdot 0\right)$$

$$= -2(-1) + 1(7) - 0(-3) = 2+7 = \boxed{9}.$$

Example Find $\begin{vmatrix} 0 & 1 & 2 \\ 0 & -4 & 7 \\ 8 & 11 & 13 \end{vmatrix}$ $\quad \begin{matrix} + \\ - \\ + \end{matrix}$

$$= +0\left|\quad\right| - 0\left|\quad\right| + 8\begin{vmatrix} 1 & 2 \\ -4 & 7 \end{vmatrix} = 8(\underbrace{7-(-8)}_{15}) = \boxed{120}$$

Geometrically:

2x2 matrix $\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} = \pm$ area of the parallelogram spanned by $v=(v_1, v_2)$ & $w=(w_1, w_2)$.

area A

$\det\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} = -A$

always $\pm$ (area).

higher dimensions

$$\begin{vmatrix} a_1 & b_1 & c_1 & \\ a_2 & b_2 & c_2 & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} = \pm \left(\begin{array}{l} \text{n-dim volume } \cancel{\text{area}} \text{ of the} \\ \text{parallelipiped spanned} \\ \text{by the vectors in rows} \\ \text{(or columns).} \end{array} \right)$$

$\text{volume} = \pm \begin{vmatrix} a & b & c \\ \vdots & \vdots & \vdots \end{vmatrix}$

In $\mathbb{R}^3$, if $\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = 0$,

what does that say about the vectors a, b, c?

Ans: a, b, c are contained in the same plane in $\mathbb{R}^3$.

Cross product of vectors

Given $v, w \in \mathbb{R}^3$, $v \times w$ is another vector in $\mathbb{R}^3$, it's defined by

$$V \times W = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \qquad + \; - \; +$$

$$= \hat{i}\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \hat{j}\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \hat{k}\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= (v_2 w_3 - v_3 w_2)\hat{i} - (v_1 w_3 - v_3 w_1)\hat{j} + (v_1 w_2 - w_1 v_2)\hat{k}.$$

Example: Calculate $(3,-1,4)\times(1,1,2)$:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 4 \\ 1 & 1 & 2 \end{vmatrix} = \hat{i}\begin{vmatrix} -1 & 4 \\ 1 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= \hat{i}\left((-1)2 - 4\cdot 1\right) - \hat{j}\left(3\cdot 2 - 4\cdot 1\right) + \hat{k}\left(3\cdot 1 - (-1)1\right)$$

$$= -6\hat{i} - 2\hat{j} + 4\hat{k} = (-6,-2,4).$$

Geometric meaning of cross product of v & $w \in \mathbb{R}^3$:

- $v \times w$ is a vector that points in a direction perpendicular to both v & w, where the direction is given by the right hand rule.

Right hand $\rightarrow$ fingers along v, curling direction toward w, thumb points in $v \times w$ direction.

- magnitude $|v \times w|$ = area of the parallelogram spanned by the two vectors

<u>Example:</u> Find the area of the parallelogram in $\mathbb{R}^3$ spanned by the vectors $\underset{v}{(1,1,-1)}$ and $\overset{w}{(3,0,2)}$.

$$v \times w = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 3 & 0 & 2 \end{vmatrix} = \hat{i}(2-0) - \hat{j}(2+3) + \hat{k}(0-3)$$

$$= 2\hat{i} - 5\hat{j} - 3\hat{k} = (2,-5,-3)$$

$$\Rightarrow \text{area} = |v \times w| = |(2,-5,3)| = \sqrt{4+25+9}$$

$$= \boxed{\sqrt{38}}\ .$$

<u>Triple Product:</u> Given 3 vectors in $\mathbb{R}^3$, v, w, z

$$\begin{vmatrix} - & v & - \\ - & w & - \\ - & z & - \end{vmatrix} = v \cdot (w \times z) = \pm \text{ volume of parallelepiped spanned by } v, w, z.$$

Why?

$$v \cdot (w \times z) = \underset{(v_1,v_2,v_3)}{v} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_1 & w_2 & w_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

Properties of Cross Product:

① $\forall\, v, w \in \mathbb{R}^3$, $\quad v \times w = -w \times v$ (anticommutative property)

② Geometric formula

$$|v \times w| = |v|\,|w|\,\sin\theta$$

③ Distributive: If $v, w, z \in \mathbb{R}^3$, then

ⓐ $(v+w)\times z = v\times z + w \times z$

ⓑ $v \times (w+z) = v\times w + v \times z$

④ Homogeneity If $v, w \in \mathbb{R}^3$, $c \in \mathbb{R}$

ⓐ $(cv)\times w = c\,(v\times w)$

ⓑ $v \times (cw) = c\,(v \times w)$.

- Cross Product is <u>not</u> associative!

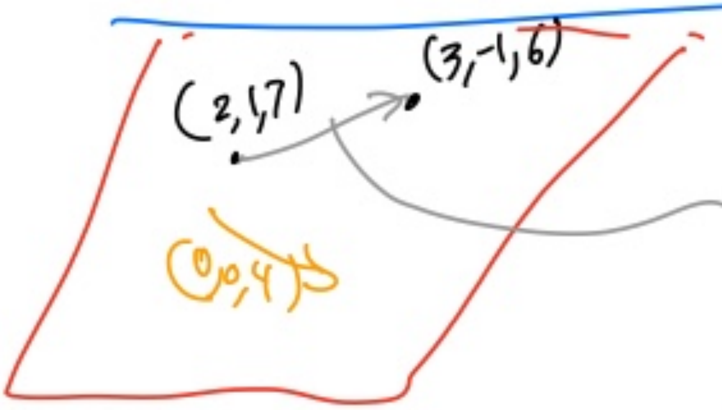
We could pick examples where

$$(v\times w)\times z \neq v \times (w \times z).$$

eg $\quad (\hat{\imath} \times \hat{\jmath}) \times \hat{\jmath} = (\hat{k}) \times \hat{\jmath} = -\hat{\imath}$.

$\hat{\imath} \times (\hat{\jmath} \times \hat{\jmath}) = \hat{\imath} \times 0 = 0$.

Example. Find the equation of the plane that contains the points $(2,1,7)$, $(3,-1,6)$ and the vector $(0,0,4)$.



Need <u>normal vector</u>

$v=(1,-2,-1)$ in plane (parallel)

$(0,0,4)$ in plane

Let $N=\begin{vmatrix} i & j & k \\ 1 & -2 & -1 \\ 0 & 0 & 4 \end{vmatrix} = \hat{i}(-8)-\hat{j}(4)+\hat{k}(0)$

$= -8\hat{i} - 4\hat{j}$

$= (-8,-4,0)$. perp to plane.

$(x_0,y_0,z_0)=(2,1,7)$

$\Rightarrow$

$-8(x-2)+-4(y-1)+0(z-7)=0$

$-8x+16-4y+4=0$

$\boxed{-8x-4y=-20}$

$-4(2x+y)=-4(5)$

$\boxed{2x+y=5}$.

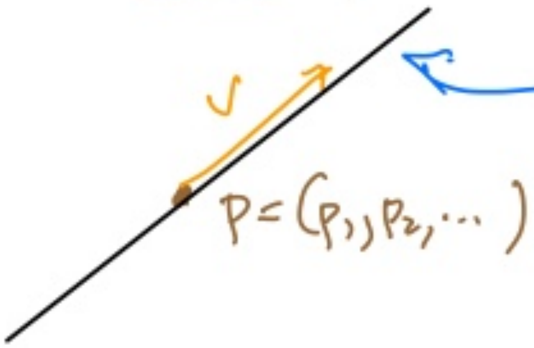
$\boxed{\text{Ex}}$ Find the equation of the plane that is parallel to the line

$$L(t) = (4+t,\ 7-t,\ 2+2t)$$

And contains the line

$$S(t) = (2-t,\ 4,\ t+1)$$

<u>Aside</u>: parametric equation of a line in $\mathbb{R}^n$.



parametric. eq.

$$L(t) = p + t\,v$$

initial point.

velocity vector
